Worksheet on Non-Linear Proportions

Gregory V. Bard

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Prerequisites

Before we start, if you were not 100% comfortable with "Direct Proportions" from that earlier worksheet, as well as the lesson "How Exponents Really Work," then you should review those two topics before attempting this topic.

I'm not joking. Be certain that you can do the "Direct Proportions" problems, and that you've read "How Exponents Really Work" before attempting this worksheet.

Overview

Below, we are going to learn that on an aircraft, the force of air resistance is proportional *to the square* of the airspeed. Likewise, we're going to learn that the volume of a scale model of a statue or piece of equipment is proportional *to the cube* of the height (or width or length.) To solve these problems, you attach a squaring symbol ^{2} or a cubing symbol ^{3} to the item that should be squared or cubed.

For example,

$$\frac{\text{Force}_A}{\text{Force}_B} = \frac{(\text{Speed}_A)^2}{(\text{Speed}_B)^2}$$

and

$$\frac{\text{Volume}_A}{\text{Volume}_B} = \frac{(\text{Height}_A)^3}{(\text{Height}_B)^3}$$

look exactly like direct proportions, except that the speed terms all are wearing squares in the first proportion, and the height terms are all wearing cubes in the second proportion. If those exponents weren't there, then these would be direct proportions. The correct exponent to use will always be stated in the problem—do not memorize those. The point is not to learn that, for example, radar signals vary in the inverse fourth power with distance to the radar station, but rather, to learn how to act upon that information correctly whenever you see it.

Each of the questions below has a crucial sentence which reveals the relationship between the variables. To make this topic easier for you, I have put the crucial sentence in italics in each of the problems. A quick vocabulary item is needed before we continue. Normally, we'll say something like "proportional to the cube" or "proportional to the 3/2s power" which means an exponent of ^{3} or of $^{3/2}$. If we say "proportional to the inverse square" that means the $^{-2}$ power, while "proportional to the inverse fourth power" means the $^{-4}$ power. Last but not least, if we say "inversely proportional" then that means the $^{-1}$ power.

Before you continue, please read each of the questions and pay particular attention to the sentence that is in italics.

Seriously, give it a try. Read each of the questions now, attempting none of the math, and try to understand the pattern of all the phrases that are in italics.

Opening Example

There was an interior decorator in my social circle who found himself vacationing in Italy. He saw there some statues of Michelangelo's *David* and imagined that many people might like such a statue in their apartments. It certainly is a beautiful statue and it is a symbol of the Renaissance that nearly anyone would recognize. Also, it didn't hurt that the statues were relatively inexpensive. In any case, each 2-foot tall poured concrete statue weighed 80 pounds. My friend's business idea was to order a set of life-size statues.

His mistake was to order a 6-foot tall version. The reason this was a mistake is because *when you make a scale model of an object, its weight grows proportional to the cube of length, width, height, radius, and so on.* Let the weight of the new statue be w, then we would write

$$\begin{array}{ccccc} \text{weight} & & \text{height}^3 & & \\ \downarrow & & \downarrow & & \\ \frac{w \text{ lbs}}{80 \text{ lbs}} & = & \frac{(6 \text{ ft})^3}{(2 \text{ ft})^3} & & \frac{\leftarrow\text{large statue}}{\leftarrow\text{small statue}} \end{array}$$

which is just like a direct proportion, except that the heights are wearing cube symbols. To solve for w is routine, and we would proceed as follows:

$$\begin{aligned} \frac{w}{80} &= \frac{6^3}{2^3} \\ \frac{w}{80} &= \frac{216}{8} \\ 8w &= (216)(80) \\ w &= (216)(80)/8 \\ w &= 2160 \end{aligned}$$

Therefore we learn that the life-sized statue weighs 2160 pounds, or more than one ton. It is phenomenally expensive to ship something *that heavy* across the Atlantic Ocean from Italy to New York City, and this resulted in an exorbitant cost for the life-sized statue. Unsurprisingly, none sold and it was a horrible financial loss. The moral of the story is that if you double, triple, or quadruple an object's size, as in making a scale model, then it is not the case that you double, triple, or quadruple the weight. As it turns out, you will see that the weight grows by a factor of 8, of 27, or of 64. Surely, we can all agree that this is a catastrophic change!

A Handy Way to Check with Units: In a non-linear proportions problem, two of the entries will have exponents, and two entries will not have exponents. The two entries with exponents must have the same units as each other. Likewise, the two entries without exponents must also have the same units as each other. In the first example, you can see that the entries with cubes are both in "feet," and the entries without cubes are both in "pounds."

Summary

Mathematically, non-linear proportions are almost the same as direct proportions. The only difference is that two of the entries in the proportions might be wearing a cube symbol, a square symbol, perhaps a square root symbol, or some exotic exponent like 3/2 or 1/3, as we'll see below. They are to be solved exactly as a direct proportion would be solved. The challenge is to read the sentence which reveals the proportionality law, and to correctly put all the numbers in the correct positions.

Questions

1. A major mistake that inexperienced web designers make is that they use too high a resolution for the photographs and other images. This makes the images larger and therefore slower to download. Let's suppose an initial photo of company headquarters was 3.5 megabytes, and takes 2.6 seconds to download. The initial photo is at a resolution of 320 pixels per inch. An intern suggests reducing the photo's resolution to only 100 pixels per inch.

   - Knowing that *the size of an image file is proportional to the square of the resolution of the original*, what do you predict for the new file size?
   - Knowing that *the download time of the image is proportional to the square of the resolution of the original*, what do you predict for the download time?

2. *The air resistance on an aircraft, when all other things are held constant, is proportional to the square of speed.* Suppose the landing gear of an airplane experience 1 ton of force if deployed at 200 mph. (Note that 1 ton is 2000 pounds.) How much force does the landing gear experience if the pilot accidentally deploys the landing gear at 650 mph?

3. You're in the business of selling ball bearings. The manager is out sick and an order comes in from a trusted customer. They say that they need a thousand ball bearings that are 1/8th inches in diameter. You fetch a sack of a thousand of those from the supply room, and weigh it for postage.

   The sack of a thousand ball bearings that are 1/8th inches diameter weighs 1.064 kilograms. The customer calls back and says that he needs another order that is 3/16ths in diameter—but you check the stockroom and you are out of those. He at least wants an estimate of how much the shipping will cost, so that you can send them when they arrive. The shipping company's website will provide the cost estimate, if you can provide the weight.

   Of course, you can't just put a sack of the 3/16s on the scale, because you don't have any in stock. How much will a sack of thousand ball bearings that are 3/16ths in diameter weigh? [Hint: you have to remember that *when you make a scale model of something, the weight grows as the cube of the height, width, length, radius, or diameter.*]

4. Let's imagine that a friend of yours has a job on campus working for a physics or chemistry professor. The faculty member is making a nice display in the hallway, showing spheres made out of various elements from the periodic table. Each sphere is exactly 1 kilogram of material. Oddly, the diameter of the spheres varies noticably depending on the materials involved. To try to understand this, you and your classmate do a quick internet search, and discover the following rule: *For any fixed mass, the diameter of a sphere made of a particular material is proportional to the inverse one-third power of the density of that material.* This rule is surprising to your friend, who is planning to test the rule by seeing what diameter it predicts for two of the spheres. Knowing that Aluminum has a density of 2.70 g/cm^3, Lead has a density of 11.34 g/cm^3, and Iron has a density of 7.874 g/cm^3, compute the diameters of the lead and aluminum spheres given that the iron sphere has a diameter of 6.24 cm.

5. It is a rough rule-of-thumb that *the cost of warehousing or storing goods is proportional to the square root (or one-halfth power) of the number of items.* Suppose a company likes to maintain a stock of one thousand toys for a particularly popular item, during the normal part of the year. They spend about $ 500 per month on warehousing costs. If they want to have 18,000 in stock for the month before Christmas, how much should they (very roughly) anticipate paying?

6. *The power of a single source of light varies as the inverse square of the distance to it.* The power of sunlight is roughly 126.4 Watts per square foot for a clear day on the earth's surface, on average. This varies, of course, with the time of the year, as the distance from the earth to the sun changes. Knowing the following distances:

- From Mercury to the Sun, at closest approach: 28,583,900 miles.
- From Earth to the Sun, on average: 92,955,800 miles.
- From Neptune to the Sun, at furthest distance: 2,829,570,000 miles.

find the power of sunlight (in Watts per square foot) on Mercury at the closest approach, and on Neptune at the furthest distance.

7. Let us imagine that you are working for a defense contracting firm designing a stealth jet-ski with a built-in belt-fed machine gun for use by Navy SEALs. The current design has a top speed of 34 knots (39.1265 mph). However, the SEALs decide they want a top speed of 45 knots (51.7850 mph), and this will require changing the engine for a more powerful one. As it turns out, in any vehicles where the motor or engine is essentially working only to counter the effects of the force of drag, *the power of the engine is proportional to the cube of the speed.* If the current engine is 290 horsepower, what will the power of the replacement engine be?

8. A geostationary[1] orbit is a popular orbit for space satellites. A geostationary orbit is always located at an altitude of 35,786 km. For comparison, the earth radius is 6371 km.

   For any particular object, *the strength of the force of the earth's gravity is proportional to the inverse square of the distance from that object to the center of the earth.* Suppose that the force on a specific astronaut on the earth's surface happens to be 778 Newtons. (The unit "Newton" is a unit of force.) How much force will the earth's gravity inflict on her when she works on repairing a satellite in geostationary orbit?

9. *The cost of enclosing an area of land with a fence is proportional to the square root of the area.* This means that if you quadruple the amount of land enclosed, the costs of the fence only double. That's a bit surprising at first glance.

   Let us consider the following scenario. Imagine a ranch in Texas that is expanding rapidly. They have one newly built enclosure that is 20,000 square feet, and it recently cost $ 9500. Can you use a non-linear proportion to estimate how much it would cost to enclose 60,000 square feet? Or 80,000 square feet? (The details of why this mathematical trick happens to be true are given in a special appendix, Page 6.)

10. Chad is 5'5" and 140 lbs, and is chatting on the internet with a cousin who is 6'4" and Chad is considering copying his cousin's workout plan and diet. Chad asks his cousin his bodyweight and the reply is 192 lbs... Chad thinks, "Wow, this guy is fatter than me, there's no way I'm going to copy his diet and workout plan!" His roommate Prathap, who is a pre-med, corrects him and reminds him that *bodyweight varies with height squared, for any fixed Body-Mass Index.*

    - So Chad is going to calculate how much his cousin would weigh if he were 5'5" instead of 6'4", and assuming they had the same BMI. What would that bodyweight be?
    - Likewise, Chad is going to calculate how much he would weigh if he were 6'4" instead of 5'5", and assuming they had the same BMI. What would that bodyweight be?

11. There has been an car accident where one car screeched to a stop. It is important to determine the speed of the car, in order to determine if the driver should be charged with a crime. The car is a Volvo C30, model year 2010, and the website of a famous car magazine says that the stopping distance at 60 mph is 143 feet. The tire-marks on the pavement measure 354 feet. Based on the information that *the speed of a car prior to coming to a stop is proportional to the square root of the length of the skid marks*, what was the speed of the car?

[1] In this orbit, a satellite will make one full lap of the earth in one day, the same time as the earth's rotation. Therefore, the satellite will appear directly over a fixed, specific location of the earth's surface forever (or almost forever.) Satellite TV services like this, because you only need to point the dish once, and it will always point to the satellite forever—unless there is a hurricane.

12. A suspicious aircraft is flying toward a military base. The radar system is tracking it. As it turns out, for any particular object, *the power of the radar signal is proportional to the inverse fourth power of the distance from the object to the radar station.* When the aircraft is 100 miles away, the signal is 25 Watts.

    (a) How strong is the signal when it is 50 miles away? 25 miles away? 10 miles away?

    (b) If the radar set is very antiquated and not very sensitive, and can only pick up a return signal of 200 Watts or more, then how far away from the airbase would the plane be before it was detected?

13. Alice and Bob are taking a vacation in near San Francisco, and as they are driving early in the morning they are surprised by some unusually heavy fog. Alice asks Bob to slow down to 35 mph, as he's going 60 mph at the moment. Bob doesn't think it would make very much difference to the stopping distance. According to the website of a very famous car magazine, the 60-mph stopping distance of Bob's car (a 2013 Toyota Camri) is 129 feet. Alice knows that *the stopping distance of a car (under standard conditions) is proportional to the square of the speed.* What would the stopping distance be if Bob took Alice's advice?

14. One of Kepler's Laws of Planetary motion is that *the period of a planet (the amount of time that it takes to do one lap of the sun) is proportional to the distance from the planet to the sun on average, raised to the* $3/2s$ *power.* The distance from the earth to the sun, on average, is 149,598,000 km, while the distance from Mars to the sun, on average, is 227,939,000 km. The length of one earth year is the length of time it takes for the earth to go around the sun; obviously that's 365.25 days. What is the length of the Martian year (an astronomer would say "the period of Mars"), in earth-days?

15. Let us suppose you have an internship with a business that makes the metal tops of trophies and other awards. These are usually electroplated—that means they are made of a cheap metal, like iron or copper, and then an ultra-thin layer of gold or silver is coated onto the metal by electrolysis. Your company specializes in silver. This is a cheap process, and almost all of the cost is in purchasing the silver nitrate. *When making a scale model of something, the surface area (and therefore the amount of silver nitrate used) is proportional to the weight raised to the 2/3rds power.*

    Their high-end silver model of lady victory ways 1 pound and uses $ 36 worth of silver nitrate. The company wants to make a special large 5 pound version for the big table in the conference room. Using a non-linear proportion, estimate how many dollars of silver nitrate that would require.

Note: Now we're going to introduce the concept of "inversely proportional." This phrase indicates the -1th power. You can actually put the -1th exponent on either side, and you will get the same answer. Of course, you do not put the -1th exponent on both sides!

16. *Volume and pressure, when temperature is kept the same, are inversely proportional.* If a certain quantity of fumes are created when an airbag goes off, and the initial pressure is 20 atmospheres, and the initial volume is 30 cubic inches, what will the volume of the airbag come to be when it reaches equilibrium? (That is to say, when the airbag reaches 1 atmosphere of pressure?)

17. Let us imagine that you are in the peace corps, helping to bring water to a desert village whose water supply has been suddenly compromised. Several jeeps and trailers have been made available to you. The limiting factor when using a jeep to tow a trailer full of emergency supplies is the weight. The current jeep-and-trailer assembly can haul 106 jugs of 5 gallons each. However, the US embassy has made available several 17.5 gallon jugs which are more suitable to this task. *The number of jugs that one jeep can haul is inversely proportional to the jug's capacity.* Using a non-linear proportion, can you predict how many jugs the jeep can haul?

18. Ned and Ted have taken summer jobs in a warehouse. They open large crates that have been delivered with a forklift, and remove the boxes inside the large crate, and put them in appropriate places in the

warehouse. Ned and Ted both have six inch pry bars (sometimes called crow bars) to assist them. One pair of crates is stubborn, and it seems that it took 45 pounds of force to get it open using the six inch pry bar. The foreman sees their struggles, and brings over "the persuader," a solid crowbar that is one yard in length. (Note, a yard is 36 inches.) Knowing that *the length of a lever and the force required to move a particular object are inversely proportional to each other*, how many pounds of force will be required to open the other stubborn crate, using "the persuader"?

19. Let us imagine that you're working as an intern at a metal processing plant. They frequently get deliveries of scrap metal, usually iron, and they dump it in one-ton piles in their warehouse prior to processing it. A ton of iron takes up about 4.5 cubic feet.

    Today, however, they are receiving a shipment of aluminum. Knowing that *the volume taken up by a ton of material is inversely proportional to its density*, can you predict the volume of a ton of aluminum? (A quick internet search reveals that the density of iron is 7.874 g/cc and the density of aluminum is 2.70 g/cc.)

20. (This is too challenging but somewhat interesting.) If I weigh 212.5 pounds in Menomonie (altitude 262 meters above sea level), then how much would I weigh on the top of Mount Everest? (Altitude = 8848 meters above sea level.) For this problem, you must also keep in mind that the radius of the earth is roughly 6371 kilometers. Knowing that *the force of gravity is proportional to the inverse square of the distance to the center of the planet*, how much would I weigh atop Mount Everest?

Appendix: Checking the "Fence Enclosure" Model

As it comes to pass, I once had a dispute with a student about this problem (# 9 of this worksheet). The student could not accept that quadrupling the enclosed area would result in only doubling the cost of the fencing materials. This got worse when I pointed out that enclosing $25\times$ as much area would result in only $5\times$ the quantity of fencing materials.

To settle the matter, I chose four specific test cases. In all the test cases before the upgrade, the area is 10,890 square feet, which is a quarter of an acre. The area after the upgrade is $10,890 \times 4 = 43,560$ square feet. The test cases are shapes from geometry: a circle, an equilateral triangle, a square, and a rectangle with a 3:1 aspect ratio. In each case, I carefully calculate how the area changes (it quadruples) and how the perimeter changes (it doubles).

I strongly encourage you to read through it all and see that I have not cheated. All the necessary geometric formulas have been provided, and therefore you do not have to remember anything from your previous geometry classes.

Without any further chatter from me, here are the test cases:

A Circle:

Geometry: The region is a circle of radius r.

Formulas: The area of a circle is $A = \pi r^2$ and the perimeter (technically called "the circumference") is $C = 2\pi r$.

Changes: In this case, the radius goes from $r = 58.8760\cdots$ feet to $r = 117.752\cdots$ feet.

Old Area: $A = \pi(58.8760)^2 = (3.14159\cdots)(3466.38\cdots) = 10,889.9\cdots$ sq ft.

New Area: $A = \pi(117.752)^2 = (3.14159\cdots)(13,865.5\cdots) = 43,559.8\cdots$ sq ft.

Area Check: $43,559.8 \div 10,889.9 = 4.00001\cdots$, so the area did quadruple.

Old Perimeter: $C = 2\pi(58.8750) = 369.922\cdots$ feet.

New Perimeter: $C = 2\pi(117.752) = 739.857\cdots$ feet.

Perimeter Check: $739.857 \div 369.922 = 2.00003\cdots$, so twice as much fencing is needed.

An Equilateral Triangle:

Geometry: The region is an equilateral triangle of side length ℓ.

Formulas: The area of an equilateral triangle is $A = \frac{\sqrt{3}}{2}\ell^2$, while its perimeter is $P = 3\ell$.

Changes: In this case, the side length goes from $\ell = 112.136\cdots$ feet to $\ell = 224.273\cdots$ feet.

Old Area: $A = \frac{\sqrt{3}}{2}(112.136)^2 = (0.866025\cdots)(12,574.4\cdots) = 10,889.8\cdots$ sq ft.

New Area: $A = \frac{\sqrt{3}}{2}(224.273)^2 = (0.866025\cdots)(50,298.3\cdots) = 43,559.6\cdots$ sq ft.

Area Check: $43,559.6 \div 10,889.8 = 4.00003\cdots$, so the area did quadruple.

Old Perimeter: $P = (3)(112.136) = 336.408$ feet.

New Perimeter: $P = (3)(224.273) = 672.819$ feet.

Perimeter Check: $672.819 \div 336.408 = 2.00000\cdots$, so twice as much fencing is needed.

A Square:

Geometry: The region is a square of side length ℓ.

Formulas: The area of a square is $A = \ell^2$, while its perimeter is $P = 4\ell$.

Changes: In this case, the side length goes from $\ell = 104.355\cdots$ feet to $\ell = 208.710\cdots$ feet.

Old Area: $A = (104.355)^2 = 10,889.9\cdots$ sq ft.

New Area: $A = (208.710)^2 = 43,559.8\cdots$ sq ft.

Area Check: $43,559.8 \div 10,889.9 = 4.00001\cdots$, so the area did quadruple.

Old Perimeter: $4(104.355) = 417.420$ feet.

New Perimeter: $4(208.710) = 834.840$ feet.

Perimeter Check: $834.840 \div 417.420 = 2$ (precisely), so twice as much fencing is needed.

A Rectangle with 3:1 Aspect Ratio:

Geometry: A rectangle that is w feet wide and $3w$ feet long is said to have an "aspect ratio" of 3:1. It is a typical rectangle.

Formulas: The area of a rectangle that is w ft wide and $3w$ ft long is $A = (w)(3w) = 3w^2$ and its perimeter is $P = w + 3w + w + 3w = 8w$.

Changes: In this case, the width goes from $60.2494\cdots$ feet to $120.498\cdots$ feet.

Old Area: $A = 3(60.2494)^2 = (3)(3629.99\cdots) = 10,889.9\cdots$ sq ft.

New Area: $A = 3(120.498)^2 = (3)(14,519.7\cdots) = 43,559.3\cdots$ sq ft.

Area Check: $43,559.3 \div 10,889.9 = 3.99997\cdots$, so the area did quadruple.

Old Perimeter: $P = 8(60.2494) = 481.995\cdots$ feet.

New Perimeter: $P = 8(120.498) = 963.984\cdots$ feet.

Perimeter Check: $963.984 \div 481.995 = 1.99998\cdots$, so twice as much fencing is needed.

Solutions

1. We start with

$$\begin{aligned} \frac{m}{3.5 \text{ megabytes}} &= \frac{(80 \text{ pixels/in})^2}{(320 \text{ pixels/in})^2} \\ (320^2)m &= (80^2)(3.5) \\ 102,400m &= 22,400 \\ m &= 0.21875 \end{aligned}$$

and thus we learn that the new image will require only 0.218750 megabytes, or 218.750 kilobytes. That's considerably smaller!

Similarly, we start with

$$\begin{aligned} \frac{T}{2.6 \text{ sec}} &= \frac{(80 \text{ pixels/in})^2}{(320 \text{ pixels/in})^2} \\ (320^2)T &= (80^2)(2.6) \\ 102,400T &= 16,640 \\ T &= 0.1625 \end{aligned}$$

and therefore we learn that the new image will require only 0.1625 seconds, or 162.5 milliseconds, to download.

This is very important, because even if you have 2–3 images that require 2.6 seconds each to download, then probably the entire webpage will take 8–10 seconds to load. Most people will mistake this for the site being down or having some trouble, and they will leave. However, the 0.1625 seconds is not going to be noticeable at all.

Note: If you are getting different proportions than I, but the correct answer, don't worry. Basically,

$$\frac{\text{thing}_1}{\text{thing}_2} = \frac{\text{junk}_1}{\text{junk}_2} \text{ is the same as } \frac{\text{thing}_2}{\text{thing}_1} = \frac{\text{junk}_2}{\text{junk}_1}$$

2. The proportion is

$$\frac{(650 \text{ mph})^2}{(200 \text{ mph})^2} = \frac{x \text{ lb-force}}{2000 \text{ lb-force}}$$

and that becomes

$$2000 \cdot \frac{650^2}{200^2} = x$$

and that comes to 21,125 pounds or 10.5625 tons.

You could also perfectly well do the entire problem in tons, instead of in pounds. If you chose that unit you would have

$$\frac{(650 \text{ mph})^2}{(200 \text{ mph})^2} = \frac{x \text{ tons-force}}{1 \text{ ton-force}}$$

and then x would come out in tons. Both methods are fine.

3. To save some time, it is best to convert 1/8th into 0.125 and 3/16th into 0.1875. That is obtained from the calculator by requesting $1 \div 8$ and $3 \div 16$. The proportion is

$$\frac{(0.1875 \text{ in})^3}{(0.125 \text{ in})^3} = \frac{x \text{ kg}}{1.064 \text{ kg}}$$

and you eventually get 3.591 kg. Isn't that amazing, that you can change from 1/8th to 3/16ths, which would be a barely visible change, and more than triple the weight?!

4. For this one, the difficulty isn't in the mathematics, but figuring out what goes where. For the iron sphere, we know both the diameter and the density, so that will take the role of the "old data." For the lead sphere, we only have the diameter, and so that will take the role of the "new data." Clearly, the density gets the exponent. We have to remember that "inverse one-third power" means an exponent of "$-1/3$," and both densities must wear that exponent. With all that in mind, we set up the non-linear proportion and solve it as follows:

$$\begin{aligned} \frac{D \text{ cm}}{6.24 \text{ cm}} &= \frac{(11.34 \text{ g/cc})^{-1/3}}{(7.874 \text{ g/cc})^{-1/3}} \\ \frac{D}{6.24} &= \frac{0.445104\cdots}{0.502652\cdots} \\ \frac{D}{6.24} &= 0.885511\cdots \\ D &= 5.52559\cdots \end{aligned}$$

Likewise, for the aluminum sphere, we perform these steps:

$$\begin{aligned} \frac{D \text{ cm}}{6.24 \text{ cm}} &= \frac{(2.70 \text{ g/cc})^{-1/3}}{(7.874 \text{ g/cc})^{-1/3}} \\ \frac{D}{6.24} &= \frac{0.718144\cdots}{0.502652\cdots} \\ \frac{D}{6.24} &= 1.42870\cdots \\ D &= 8.91514\cdots \end{aligned}$$

5. Each of the quantities of toys should have a "1/2" power on it as an exponent.

$$\begin{aligned} \frac{(18,000 \text{ toys})^{1/2}}{(1000 \text{ toys})^{1/2}} &= \frac{\$x}{\$500} \\ \frac{134.164\cdots}{31.6227\cdots} &= \frac{x}{500} \\ 4.24266\cdots &= \frac{x}{500} \\ (4.24266\cdots)(500) &= x \\ 2121.33\cdots &= x \end{aligned}$$

And so the rough approximation is that they would pay \$ 2121.33.

6. This solution is broken into two parts, one for Mercury and one for Neptune.

- For Mercury, we start with

$$\begin{aligned}
\frac{(28,583,900 \text{ km})^{-2}}{(92,955,800 \text{ km})^{-2}} &= \frac{x \text{ W/sq ft}}{126.4 \text{ W/sq ft}} \\
(126.4)(28,583,900)^{-2} &= x(92,955,800)^{-2} \\
(126.4)(1.22393\cdots \times 10^{-15}) &= x(1.15730\cdots \times 10^{-16}) \\
\frac{(126.4)(1.22393\cdots \times 10^{-15})}{1.15730\cdots \times 10^{-16}} &= x \\
1336.77\cdots &= x
\end{aligned}$$

  which is incredible! The earth's sunlight (on a clear day) is like having two sixty-watt lightbulbs over each square foot. Mercury would be like having $1336.77\cdots/60 = 22.2795\cdots$ lightbulbs over each square foot! No wonder Mercury is somewhat warm. The surface temperature at the equator can reach 800°F or 427°C.

- For Neptune, the solution is almost the same

$$\begin{aligned}
\frac{(2,829,570,000 \text{ km})^{-2}}{(92,955,800 \text{ km})^{-2}} &= \frac{x \text{ W/sq ft}}{126.4 \text{ W/sq ft}} \\
(126.4)(2,829,570,000)^{-2} &= x(92,955,800)^{-2} \\
(126.4)(1.24899\cdots \times 10^{-19}) &= x(1.15730\cdots \times 10^{-16}) \\
\frac{(126.4)(1.24899\cdots \times 10^{-19})}{1.15730\cdots \times 10^{-16}} &= x \\
0.136414\cdots &= x
\end{aligned}$$

  which is shocking in the other direction. The lightbulbs would have to be spaced very far apart to provide that little light. As it turns out, they'd have to be evenly spaced about 21 feet from each other, but were not asked that and so will not consider it now. But in any case, this is why Neptune is somewhat cold. This is why Neptune is called an "ice giant."

7. You can choose to do this problem either in knots or in mph, as you prefer. If you use knots, then you would have

$$\frac{(45 \text{ knots})^3}{(34 \text{ knots})^3} = \frac{h}{290 \text{ hp}}$$

   which becomes

$$\frac{91,125}{39,304} = \frac{h}{290}$$

   so that $h = (290)(91,125)/39,304 = 672.355\cdots$ horsepower.

   Alternatively, if you prefer mph, then you would write

$$\frac{(51.7850 \text{ mph})^3}{(39.1265 \text{ mph})^3} = \frac{h}{290 \text{ hp}}$$

   which becomes

$$\frac{138,871.\cdots}{59,898.\cdots} = \frac{h}{290}$$

   so that $h = (138,871.\cdots)(290)/59,898.\cdots = 672.352\cdots$ horsepower.

   As you can see, both approaches yield essentially the same answer.

8. Naturally the astronaut when on the earth's surface is 6371 km from the center of the earth, and when at an altitude of 35,786 km, is at an altitude of 6371 km+35,786 km. The phrase "proportional to the inverse square of distance" means that all the distances will be raised to a $^{-2}$ power. Now we have

$$\begin{aligned}
\frac{(35,786\text{ km} + 6371\text{ km})^{-2}}{(6371\text{ km})^{-2}} &= \frac{x\text{ N}}{778\text{ N}} \\
\frac{42,157^{-2}}{6371^{-2}} &= \frac{x}{778} \\
(778)(42,157)^{-2} &= x(6371)^{-2} \\
(778)(5.62678\cdots \times 10^{-10}) &= x(2.46368 \times 10^{-8}) \\
\frac{(778)(5.62678\cdots \times 10^{-10})}{(2.46368 \times 10^{-8})} &= x \\
17.7686\cdots &= x
\end{aligned}$$

Thus we learn that the force upon her will be 17.7686 N or roughly 4 pounds of force. Her initial weight of 778 N is actually 175 pounds of force.

9. As we saw by the sentence in italics, the square root will go on the areas. Then we can set up the non-linear proportion as follows:

$$\begin{aligned}
\frac{c\text{ dollars}}{9500\text{ dollars}} &= \frac{\sqrt{60,000\text{ sq ft}}}{\sqrt{20,000\text{ sq ft}}} \\
\frac{c}{9500} &= \frac{244.948\cdots}{141.421\cdots} \\
\frac{c}{9500} &= 1.73205\cdots \\
c &= (9500)(1.73205\cdots) \\
c &= 16,454.4\cdots \approx \$\ 16,454
\end{aligned}$$

$$\begin{aligned}
\frac{c\text{ dollars}}{9500\text{ dollars}} &= \frac{\sqrt{80,000\text{ sq ft}}}{\sqrt{20,000\text{ sq ft}}} \\
\frac{c}{9500} &= \frac{282.842\cdots}{141.421\cdots} \\
\frac{c}{9500} &= 2 \\
c &= (9500)(2) \\
c &= 19,000
\end{aligned}$$

Therefore, we see that the 60,000 sq ft enclosure would cost \$ 16,454 and the 80,000 sq ft enclosure would cost \$ 19,000. It is interesting how quadrupling the area enclosed only doubles the cost of the materials. This point is explored deeper in a special appendix, on Page 6

10. First, it is very important to convert all the heights into inches. The 5'5" height becomes $5 \times 12 + 5 = 65$ inches and the 6'4" height becomes $6 \times 12 + 4 = 76$.

- Then we can set the first bullet up as follows

$$\frac{(65\text{ in})^2}{(76\text{ in})^2} = \frac{x\text{ lbs}}{192\text{ lbs}}$$

and therefore

$$x = 192\frac{65^2}{76^2} = 192\frac{4225}{5776} = 140.443\cdots$$

and we conclude that the cousin would be 140.443 pounds if he were 5'5" instead of 6'4".

- The second bullet would be

$$\frac{(76\text{ in})^2}{(65\text{ in})^2} = \frac{x\text{ lbs}}{140\text{ lbs}}$$

and therefore

$$x = 140\frac{76^2}{65^2} = 140\frac{5776}{4225} = 191.394\cdots$$

and we conclude that Chad would be 191.394 pounds if he were 6'4" instead of 5'5".

11. We start with

$$\begin{aligned} \frac{s}{60\text{ mph}} &= \frac{\sqrt{354\text{ ft}}}{\sqrt{143\text{ ft}}} \\ \frac{s}{60} &= \frac{18.8148\cdots}{11.9582\cdots} \\ (11.9582\cdots)(s) &= (60)(18.8148\cdots) \\ (11.9582\cdots)(s) &= (60)(18.8148\cdots) \\ (11.9582\cdots)(s) &= 1128.89\cdots \\ s &= 94.4027\cdots \end{aligned}$$

Therefore we can conclude that the Volvo was going 94.4027 miles per hour. (In reality, we'd report 94–95 mph, because we do not actually have that much precision on the input values.)

12. This problem is in two parts.

(a) For the 50 miles away problem we have

$$\frac{(50\text{ miles})^{-4}}{(100\text{ miles})^{-4}} = \frac{x\text{ Watts}}{25\text{ Watts}}$$

which becomes

$$\begin{aligned} (x)(100^{-4}) &= (50^{-4})(25) \\ (x)(10^{-8}) &= (1.6\times 10^{-7})(25) \\ x &= \frac{25\times 1.6\times 10^{-7}}{10^{-8}} \\ x &= 400 \end{aligned}$$

and so the return would be 400 Watts. The way to solve the 25-mile and 10-mile inquiries is similar, and would come to 6400 Watts and 250,000 Watts

(b) The initial setup is

$$\frac{25\text{ Watts}}{200\text{ Watts}} = \frac{(100\text{ miles})^{-4}}{(x\text{ miles})^{-4}}$$

and we cross multiply to obtain

$$\begin{aligned}(25)(x^{-4}) &= (200)(100^{-4})\\ x^{-4} &= \frac{(200)(10^{-8})}{25}\\ x^{-4} &= 8\times 10^{-8}\\ x &= (8\times 10^{-8})^{-1/4}\\ x &= 59.4603\cdots\end{aligned}$$

and therefore we see that the aircraft would have to be 59.4603 miles away before it was first detected. If the aircraft has a 61-mile range missile, then the airbase is in serious trouble.

13. Here is how you solve this one:

$$\begin{aligned}\frac{D}{129\text{ ft}} &= \frac{(35\text{ mph})^2}{(60\text{ mph})^2}\\ (D)(60^2) &= (129)(35^2)\\ 3600D &= 158,025\\ D &= 158,025/3600 = 43.8958\cdots\end{aligned}$$

As you can see, the stopping distance of the car would be about 43.9 feet instead of 129 feet. The remainder of this discussion is not part of the problem, but is informative. If Bob suddenly notices an obstacle such as a broken-down vehicle in his lane at a distance of 100 feet away, then at the 35 mph speed he will have 100-43.9 = 56.1 feet to spare. However, if he is going 60 mph, then it will not be possible for him to stop prior to impact with the broken-down vehicle.

14. We start with

$$\frac{(149,598,000\text{ km})^{3/2}}{(227,939,000\text{ km})^{3/2}} = \frac{365.25\text{ days}}{x\text{ days}}$$

and then we cross multiply to get

$$\begin{aligned}x(149,598,000)^{3/2} &= (365.25)(227,939,000)^{3/2}\\ x(1.82973\times 10^{12}) &= (365.25)(3.44134\times 10^{12})\\ x &= \frac{(365.25)(3.44134\times 10^{12})}{1.82973\times 10^{12}}\\ x &= 686.489\cdots\end{aligned}$$

15. This one is straight-forward to solve.

$$\begin{aligned}\frac{D\text{ dollars}}{36\text{ dollars}} &= \frac{(5lbs)^{2/3}}{(1lb)^{2/3}}\\ \frac{D}{36} &= \frac{5^{2/3}}{1^{2/3}}\\ \frac{D}{36} &= \frac{2.92401\cdots}{1}\\ D &= (36)(2.92401\cdots)\\ D &= 105.264\cdots \approx \$\ 105.26 - -\$\ 105.27\end{aligned}$$

Therefore, we can conclude that \$ 105.26 or \$ 105.27 worth of silver nitrate would be required. Now if you are curious, on August 1st, 2014, I looked up the price of silver, which was \$ 652.66 per kilogram. That means the 5 pound statue, if made of solid silver, would cost \$ 1480.20 just for the silver itself, before any casting of the statue or other jeweler's work. As you can see, electroplating is much cheaper.

16. The proportion can be written in two ways. Either

$$\frac{20 \text{ atm}}{1 \text{ atm}} = \frac{(30 \text{ cu in})^{-1}}{(x \text{ cu in})^{-1}}$$

or alternatively

$$\frac{(20 \text{ atm})^{-1}}{(1 \text{ atm})^{-1}} = \frac{30 \text{ cu in}}{x \text{ cu in}}$$

If you choose the first approach, then solve the problem this way:

$$\begin{aligned} \frac{20}{1} &= \frac{30^{-1}}{x^{-1}} \\ 20x^{-1} &= 30^{-1} \\ 20x^{-1} &= 0.03\overline{3} \\ x^{-1} &= 0.0016\overline{6} \\ x &= 600 \end{aligned}$$

and so we learn that the airbag will expand to be 600 cubic inches, considerably larger than 30 cubic inches!

If you choose the second approach, which I prefer because we don't have an exponent on the x, then solve the problem this way:

$$\begin{aligned} \frac{20^{-1}}{1^{-1}} &= \frac{30}{x} \\ 20^{-1}x &= 30 \\ 0.05x &= 30 \\ x &= 600 \end{aligned}$$

which brought us considerably faster to our answer of 600 cubic inches. As I noted in the answer to the previous problem, I think these "inverse proportion" problems are much easier when x does not wear a $^{-1}$ power.

Either way, we have $x = 600$ cubic inches when the airbag equalizes to normal pressure (1 atmosphere).

17. In this problem, we see "inversely proportional" and thus we know we'll need the "negative one" power. We should put the $^{-1}$ symbol on the capacities of the jugs so that we don't have a $^{-1}$ on a variable. In this case, we obtain the following calculation:

$$\begin{aligned} \frac{J \text{ jugs}}{106 \text{ jugs}} &= \frac{(17.5 \text{ gal})^{-1}}{(5 \text{ gal})^{-1}} \\ \frac{J}{106} &= \frac{17.5^{-1}}{5^{-1}} \\ \frac{J}{106} &= \frac{0.0571428\cdots}{0.2} \\ 0.2J &= (0.571428\cdots)(106) \\ 0.2J &= 6.05714\cdots \\ J &= 30.2857\cdots \end{aligned}$$

We should predict "between 30 and 31 jugs," or we can be safe and say "30 jugs."

18. The proportion can be written in two ways. Either

$$\frac{(45 \text{ lb-force})^{-1}}{(x \text{ lb-force})^{-1}} = \frac{6 \text{ inches}}{36 \text{ inches}}$$

or alternatively

$$\frac{45 \text{ lb-force}}{x \text{ lb-force}} = \frac{(6 \text{ inches})^{-1}}{(36 \text{ inches})^{-1}}$$

because the problem said "inversely proportional to each other." The solutions in each case are somewhat different.

If you choose the first approach, then

$$\begin{aligned}
\frac{45^{-1}}{x^{-1}} &= \frac{6}{36} \\
(36)(45^{-1}) &= 6x^{-1} \\
(36)(0.02\overline{2}) &= 6x^{-1} \\
0.8 &= 6x^{-1} \\
0.8/6 &= x^{-1} \\
6/0.8 &= x \\
7.5 &= x
\end{aligned}$$

which means that 7.5 pounds of force will be required.

If you choose the second approach, then

$$\begin{aligned}
\frac{45}{x} &= \frac{6^{-1}}{36^{-1}} \\
(36^{-1})(45) &= 6^{-1}x \\
(0.027\overline{7})(45) &= (0.16\overline{6})x \\
1.25 &= (0.16\overline{6})x \\
1.25/0.16\overline{6} &= x \\
7.5 &= x
\end{aligned}$$

and of course, we get the same answer.

Now while the two methods are equivalent, my personal experience is that students find it easier to understand, and are less likely to make a mistake, if x never wears a $^{-1}$ exponent.

19. As usual, we put the $^{-1}$th power on the known attribute, rather than the unknown attribute.

$$\begin{aligned}
\frac{(2.70 \text{ g/cc})^{-1}}{(7.874 \text{ g/cc})^{-1}} &= \frac{V \text{ cu ft}}{4.5 \text{ cu ft}} \\
\frac{(2.70)^{-1}}{(7.874)^{-1}} &= \frac{V}{4.5} \\
\frac{0.370370\cdots}{0.127000\cdots} &= \frac{V}{4.5} \\
2.91629\cdots &= \frac{V}{4.5} \\
13.1233\cdots &= V
\end{aligned}$$

Our best prediction is 13.1233 cubic feet, and on a math test, this is what you would write. However, since input data was approximate, namely "about a ton" and "about 4.5 cubic feet," we should give our answer approximately also. The rules of significant figures tell us that we should have 2 figures at this point, so we would report "about 13 cubic feet."

20. Let x be my weight atop Mount Everest. From where I stand in Menomonie, it is 262 m to sea level, and another 6,371,000 meters to the center of the earth. However, atop Mount Everest, it would be 8848 meters to sea level, and then 6,371,000 meters to the center of the earth. Therefore, we write the following direct proportion

$$\frac{x}{212.5 \text{ lbs}} = \frac{(6,371,000 \text{ m} + 8848 \text{ m})^{-2}}{(6,371,000 \text{ m} + 262 \text{ m})^{-2}}$$

and cross-multiplication brings us to

$$\begin{aligned} x(6,371,000 + 262)^{-2} &= (212.5)(6,371,000 + 8848)^{-2} \\ x(6,371,262)^{-2} &= (212.5)(6,379,848)^{-2} \\ x(2.46348\cdots \times 10^{-14}) &= (212.5)(2.45685\cdots \times 10^{-14}) \\ x(2.46348\cdots \times 10^{-14}) &= (5.22080\cdots \times 10^{-12}) \\ x &= \frac{5.22080\cdots \times 10^{-12}}{2.46348\cdots \times 10^{-14}} \\ x &= 211.928\cdots \end{aligned}$$

Thus we learn that while gravity is indeed weaker atop Mount Everest than it is in Menomonie, Wisconsin, because of the distance to the center of the earth, it isn't a huge difference. On the other hand, an high-end digital bathroom scale would be able to detect the difference, as it would read 212.0 instead of 212.5.