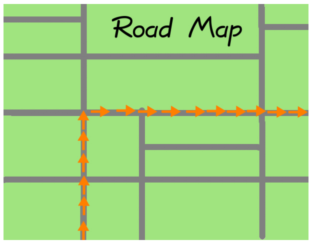
Module 3.8: Very Optional: Rigorously Deriving the Laws of Logarithms



In the next module, we're going to learn how to use the laws of logarithms. The logarithm is an exceptionally-handy mathematical function, but it can be very hard to understand why its several laws actually work. Personally, I think it is phenomenally useful to read the rigorous and precise derivation of the laws. However, after a few years of teaching, I've discovered that the students who are interested in this material often are few in number. Therefore, this is isolated here for the intrepid student who wants to know "why," but I commend the material to all students.

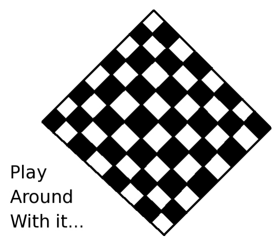
Suppose there existed a function $f(x)$ that had the property

$$f(xy) = f(x) + f(y)$$

What would we be able to do with that? What could we learn from it? Such a function does exist: it is the logarithm. In fact, there are three of them in regular use. One called the common logarithm and the other called the natural logarithm are buttons on your calculator. The third, the binary logarithm, is very important in both computer science and radioactivity, but needs to be calculated indirectly on most calculators.

We will begin assuming nothing about logarithms, with the hope of slowly building up our knowledge of this important tool. We will discover its properties, as the first users of it would have done themselves, roughly 400 years ago. For simplicity, we will focus on the common logarithm.

There are lots of laws of logarithms, but I've organized them into "the five major laws" and "the four minor laws," because some are more important than others. Most books just number them.



3-8-1

If $f(x)$ is a function such that $f(xy) = f(x) + f(y)$ then what are the following?

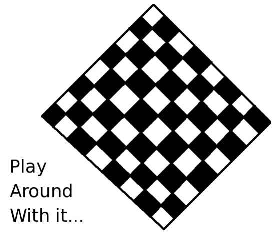
- What is $f(6x)$? [Answer: $f(6) + f(x)$.]
- What is $f(6xy)$? [Answer: $f(6) + f(x) + f(y)$.]

So we know that $x^2 = xx$ and according to the previous box, we can conclude $f(x^2) = f(x) + f(x)$. Yet this means that $f(x^2) = 2f(x)$. Interesting.

Likewise, $x^3 = x^2 x$ and according to the previous box we can conclude $f(x^3) = f(x^2) + f(x)$. Yet, we know that $f(x^2) = 2f(x)$ so we have $f(x^3) = 2f(x) + f(x)$. So now we have that $f(x^3) = 3f(x)$. Perhaps there is a pattern here!

One last example. Surely $x^4 = x^3 x$ and $f(x^4) = f(x^3) + f(x)$ likewise; we know that $f(x^3) = 3f(x)$ so we have $f(x^4) = 3f(x) + f(x)$. So now we have that $f(x^4) = 4f(x)$. The pattern continues!

With a little more work, we can prove that $f(x^n) = nf(x)$, for any positive integer n. By positive integer, we mean 1, 2, 3, 4, . . . and so on, forever. This will be a useful rule—and I call it the second major law of logarithms. The first law of logarithms is that $f(xy) = f(x) + f(y)$.



3-8-2

If $f(x)$ is a function such that $f(xy) = f(x) + f(y)$ then what are the following?

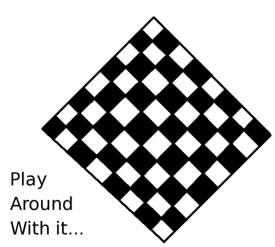
- What is $f(x^5)$? [Answer: $5f(x)$.]
- What is $f(3x^2)$? [Answer: $f(3) + 2f(x)$.]
- What is $f(x^2y^3)$? [Answer: $2f(x) + 3f(y)$.]
- What is $f(6x^4y^5)$? [Answer: $6 + 4f(x) + 5f(y)$.]

Now it turns out this is true for numbers n that are not only integers, but that are positive real numbers, or even negative real numbers. You can take my word on it if you like, or you can read the next four boxes, if you really are a lover of mathematics.

What about $f(\sqrt{x})$ or $f(\sqrt[3]{x})$? The derivations are really similar, so I'm going to do them side by side, so you can see what is different and what is the same.

$$\begin{array}{rcl|rcl} (\sqrt{x})^2 & = & x & (\sqrt[3]{x})^3 & = & x \\ \sqrt{x}\sqrt{x} & = & x & \sqrt[3]{x}\sqrt[3]{x}\sqrt[3]{x} & = & x \\ f(\sqrt{x}\sqrt{x}) & = & f(x) & f(\sqrt[3]{x}\sqrt[3]{x}\sqrt[3]{x}) & = & f(x) \\ f(\sqrt{x}) + f(\sqrt{x}) & = & f(x) & f(\sqrt[3]{x}) + f(\sqrt[3]{x}) + f(\sqrt[3]{x}) & = & f(x) \\ 2f(\sqrt{x}) & = & f(x) & 3f(\sqrt[3]{x}) & = & f(x) \\ f(\sqrt{x}) & = & f(x)/2 & f(\sqrt[3]{x}) & = & f(x)/3 \end{array}$$

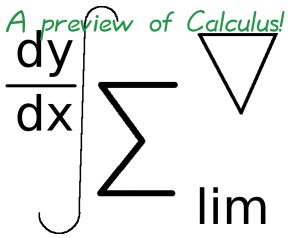
With slightly more effort, we could prove $f(\sqrt[n]{x}) = f(x)/n$. This I call the third minor law of logarithms, because if you know the laws of exponents, it can be considered a special case of $f(x^{1/n}) = \frac{1}{n} f(x)$. A list of all the laws will be found on Page 503.



3-8-3

If $f(x)$ is a function such that $f(xy) = f(x) + f(y)$ then what are the following?

- What is $f(\sqrt{3})$? [Answer: $f(3)/2$.]
- What is $f(\sqrt[7]{x})$? [Answer: $f(x)/7$.]
- What is $f(\sqrt{xy})$? [Answer: $f(x)/2 + f(y)/2$.]



Finally, we know how to handle powers in general. We can write

$$\begin{aligned} f(x^{a/c}) &= f((x^{1/c})^a) \\ &= af(x^{1/c}) \\ &= af(\sqrt[c]{x}) \\ &= af(x)/c \\ &= (a/c)f(x) \end{aligned}$$

and therefore handle any rational number in the exponent. This means the exponent can be any integer or fraction (so long as it is positive) but we do not yet know how to handle $f(x^{\sqrt{2}})$ or $f(x^{\pi})$, because those aren't rational exponents.

On the other hand, there are two reasons you should not worry about this. The first is that fixing this requires "The Squeeze Theorem" from Calculus, which is beyond the scope of this book, and the second is that your calculator knows what to do with these, at least accurate to eight decimal places.

What about $f(1)$?

$$\begin{aligned} 1 \times a &= a \\ f(1 \times a) &= f(a) \\ f(1) + f(a) &= f(a) \\ f(1) &= f(a) - f(a) \\ f(1) &= 0 \end{aligned}$$

Thus, any function that satisfies the requirements of f will have $f(1) = 0$. This I call the first minor law of logarithms.

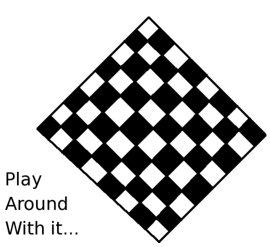
Let's consider the fact that $(a)(1/a) = 1$. If we take f of both sides we get

$$f(a \times 1/a) = f(a) + f(1/a) = f(1)$$

but since $f(1) = 0$, this means that $f(a) + f(1/a) = 0$, or finally

$$f(1/a) = -f(a)$$

and this I call the second minor law of logarithms.



3-8-4

If $f(x)$ is a function such that $f(xy) = f(x) + f(y)$ then what are the following?

- What is $f(1/ax)$? [Answer: $-f(a) - f(x)$.]
- What is $f(1/x^2)$? [Answer: $-2f(x)$.]
- What is $f(\frac{1}{3x^2y^3})$? [Answer: $-f(3) - 2f(x) - 3f(y)$.]

Now in the previous derivation involving $f(a^n)$, we assumed that the exponent was positive. Can it work for the negative exponents too?

If n is negative, then let $c = -n$, and so c will be positive. We can write

$$a^n = a^{(-1 \times -n)} = (a^{-1})^{-n} = (1/a)^{-n} = (1/a)^c$$

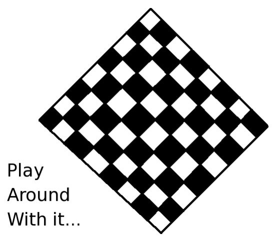
which is just a transformation, using a few of the laws of exponents.

Then, when we want to learn $f(a^n)$, we can instead find $f((1/a)^c)$. Since c is positive, we can use the second major law of logarithms, and get that

$$f(a^n) = f((1/a)^c) = cf(1/a) = c(-f(a)) = -cf(a)$$

where the very last step used our second minor law of logarithms. Finally, since $c = -n$ then also $-c = n$, and so we have $f(a^n) = nf(a)$ even when n was negative.

Now, any rational exponent is acceptable, whether positive or negative.



3-8-5

If $f(x)$ is a function such that $f(xy) = f(x) + f(y)$ then what are the following?

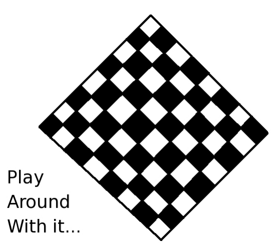
- What is $f(x^{-7})$? [Answer: $-7f(x)$.]
- What is $f(3z^{-2})$? [Answer: $f(3) - 2f(z)$.]
- What is $f(x^{-3}y^4z^{-1})$? [Answer: $-3f(x) + 4f(y) - f(z)$.]

We're almost done! Thank you for staying with me! Now let's see if we can finally handle division. If someone gives you a/c, it is at times useful to remember that $a/c = a(c^{-1})$. Then you could write

$$f(a/c) = f(a(c^{-1})) = f(a) + f(c^{-1}) = f(a) + (-1)f(c) = f(a) - f(c)$$

This is the third major law of logarithms.

Perhaps it is elegant that since multiplication turns into addition, it is only natural that division turns into subtraction. A further extension of this symmetry is that, as exponents turn into a multiple of f, likewise taking roots turns into a division of f.



3-8-6

If $f(x)$ is a function such that $f(xy) = f(x) + f(y)$ then what are the following?

- What is $f(3/2)$? [Answer: $f(3) - f(2)$.]
- What is $f(x/3)$? [Answer: $f(x) - f(3)$.]
- What is $f(19/xy)$? [Answer: $f(19) - f(x) - f(y)$.]

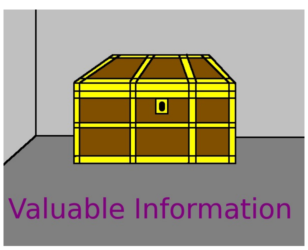
With regards to our hypothetical function f, what would happen if one were to assume $f(10) = 1$, in particular? What would $f(100)$, $f(1000)$, and $f(10,000)$ come out to? How about $f(100,000)$ or $f(1,000,000)$? Well, if you remember scientific notation, then you can calculate:

$$\begin{aligned}
f(10) = f(10^1) &= 1f(10) &= 1 \times 1 = 1\\
f(100) = f(10^2) &= 2f(10) &= 2 \times 1 = 2\\
f(1000) = f(10^3) &= 3f(10) &= 3 \times 1 = 3\\
f(10,000) = f(10^4) &= 4f(10) &= 4 \times 1 = 4\\
f(100,000) = f(10^5) &= 5f(10) &= 5 \times 1 = 5\\
f(1,000,000) = f(10^6) &= 6f(10) &= 6 \times 1 = 6
\end{aligned}$$

and so we see the basic relationship.

Up to this point, we were talking about all functions in the infinite family with the property that $f(ac) = f(a) + f(c)$. Now, we restrict ourselves to one where we assume $f(10) = 1$. Then, we learn that our f is the "opposite" or "antifunction," called "inverse function", of 10^x. This particular f is called the *common logarithm*. The role of 10 in this case is called "the base of the logarithm."

We'd say that $f(x)$ is the logarithm base b if and only if $f(b) = 1$.



A logarithm can actually have any base. It is just that 10 is really convenient, and frequently used. That's why it is understandable that the logarithm base 10 is called "the common logarithm."

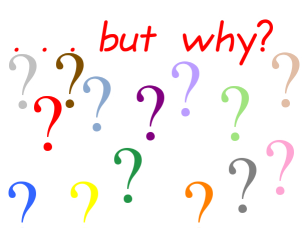
Other choices that are commonly used are 2 or e, where e is the special number

$$e = 2.718281828\ldots$$

that comes up in many branches of mathematics. We'll learn a lot about e starting on Page 519.

If you're using an unusual base, such as 5 or 7, you would write $\log_5 x$ or $\log_7 y$ to signify that base. Without a subscript, it is assumed to be the common logarithm. In other words, $\log x$ means $\log_{10} x$.

There are two more major laws of logarithms that we should look at. Together, these represent the idea that the common logarithm and 10^x are inverse functions, antifunctions, opposites, or antidotes for each other. This might or might not be a concept that you've seen before, so we'll explore it now.



We're going to assume that $f(10) = 1$ for this box, so we'll write $\log x$ to indicate that we're using base 10, or the common logarithm. First, let's see that log will undo 10^x.

$$\log 10^x = x \log 10 = x(1) = x$$

This explains the cause of the pattern we observed three boxes ago. For a base other than 10, we would want to write

$$\log_b b^x = x \log_b b = x(1) = x$$

producing essentially the same rule. This is the fourth major law of logarithms.

It turns out that to prove the reverse, that

$$10^{\log x} = x$$

will be slightly harder. To do this, we're going to need the rule that if

$$\text{if } \log(\text{junk}) = \log(\text{stuff}) \text{ then } \text{junk} = \text{stuff}$$

which will be the fourth minor law of logarithms.

This is one of those facts that a non-mathematician would just assume is obvious. However, we don't actually have the right to assume it.

This proof uses several of the laws we've already derived. Therefore, it might be a fun exercise for you to go line by line through it, and see if you can figure out why each step is justifiable.

$$\begin{aligned}
\log(\text{junk}) &= \log(\text{stuff}) \\
\log(\text{junk}) - \log(\text{stuff}) &= 0 \\
\log\left(\frac{\text{junk}}{\text{stuff}}\right) &= 0 \\
\left(\frac{\text{junk}}{\text{stuff}}\right) &= 1 \\
\text{junk} &= (1)(\text{stuff}) \\
\text{junk} &= \text{stuff}
\end{aligned}$$

As you can see, we've now proven

$$\text{if } \log(\text{junk}) = \log(\text{stuff}) \text{ then } \text{junk} = \text{stuff}$$

which will be the fourth minor law of logarithms.

Now we're ready for a really interesting proof. The fifth major law of logarithms $10^{\log x} = x$, will now follow from the fourth major and fourth minor law of logarithms. We'll start with the fourth major law:

$$\log 10^x = x$$

and substitute $x = \log y$ to obtain the rather terrifying equation

$$\log 10^{\log y} = \log y$$

Now we are in a position to use the fourth minor law of logarithms. Here, "junk" will be $10^{\log y}$ and "stuff" will be just y by itself. We have now that "junk = stuff" which becomes

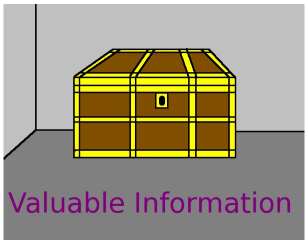
$$10^{\log y} = y$$

exactly as we wanted. This is the fifth major law of logarithms.

We should note that this fifth major law of logarithms also applies for $b \neq 10$ as well. We would say

$$b^{\log_b y} = y$$

instead. Or if you prefer $b^{f(y)} = y$ where $f(b) = 1$.



If $f(x)$ is a function such that

$$f(xy) = f(x) + f(y)$$

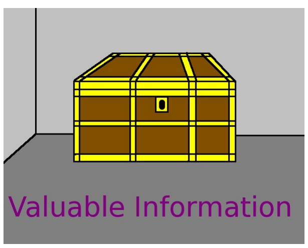
then we say that $f(x)$ is a *logarithm*. If $f(b) = 1$ then we say that it is the logarithm base b. Usually $b = 10$.

The five major laws of logarithms are

1. $f(xy) = f(x) + f(y)$ which means $\log(x + y) = \log x + \log y$.
2. $f(x^n) = nf(x)$ which means $\log(x^n) = n \log x$.
3. $f(x/y) = f(x) - f(y)$ which means $\log(x/y) = \log x - \log y$.
4. For the base b being used, $f(b^x) = x$, which means $\log 10^x = x$.
5. For the base b being used, $b^{f(x)} = x$, which means $10^{\log x} = x$.

Note: For $b \neq 10$, Number 4 is rewritten $\log_b b^x = x$.

Note: For $b \neq 10$, Number 5 is rewritten $b^{log_b x} = x$.

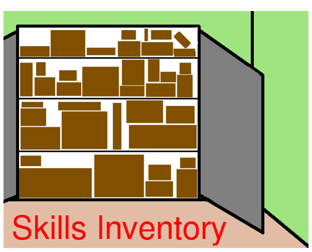


Continuing with the previous box, the four minor laws are

1. $f(1) = 0$ which means $\log 1 = 0$.
2. $f(1/a) = -f(a)$ which means $\log(1/a) = -\log a$.
3. $f(\sqrt[n]{x}) = f(x)/n$ which means $\log \sqrt[n]{x} = \frac{1}{n} \log x$.
4. If $f(x) = f(y)$ then $x = y$, which means that

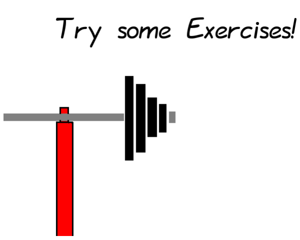
$$\text{if } \log x = \log y \text{ then } x = y$$

It is useful to note that all nine laws follow directly from the definition of a logarithm.



We have learned the following skills in this module:

- The laws of logarithms.
- Their derivations.
- As well as the vocabulary terms: common logarithm, and logarithm.



Coming Soon!