Worksheet on Solving Problems Using Logarithms

Math-123: Finite & Financial Mathematics

February 18, 2017

Questions

1. You've already learned how to compute the rate in a continuous compound interest problem. Please verify that skill by doing (or redoing) checkerboard Problem 3-9-4 and 3-9-8, in "Module 3.9: The Number e and Continuously Compounded Interest."
2. You've already learned how to compute how long a debt or an investment (with continuously compounding interest) will take to reach a certain amount. Please verify that skill by doing (or redoing) checkerboard Problem 3-9-7, in "Module 3.9: The Number e and Continuously Compounded Interest."
3. You've already learned how to compute the doubling time of an investment (with continuously compounding interest.) Please verify that skill by doing (or redoing) checkerboard Problem 3-9-12 and 3-9-13, in "Module 3.9: The Number e and Continuously Compounded Interest."
4. You've already learned how to compute how long an investment (with ordinary compound interest) will take to reach a certain amount. Please verify that skill by doing (or redoing) checkerboard Problems 3-8-12, 3-8-13, 3-8-14, 3-8-15, 3-8-16, 3-8-17, in "Module 3.8: What is a Logarithm." (By the way, the fact that there are 5 separate checkerboards for this one solitary topic might be a hint that this particular question, and the skill that it verifies, is very important.)
5. With the previous two problems in mind, compute exactly how long it would take for an investment at 7.8% compounded weekly to double in value.

Note: Among the banking and finance communities, there is a very famous "trick" called the "Rule of 72" and all the finance textbooks include it. If you take 72, and divide it by the interest rate of an investment, this is supposedly an approximation of the doubling time of that investment.

For example, at a rate of 8%, the Rule of 72 predicts that the doubling time should be nine years, because $72/8 = 9$. This rule harkens back to the time before scientific calculators became affordable in the early 1980s, and so exact computations were not something one could just do during a meeting. Let's take a look now and see just how well the Rule of 72 predicts a doubling time.

6. What does the Rule of 72 predict will be the doubling time of the investment from the previous problem? What is the residual error of that prediction? The relative error?

7. Compute precisely the tripling time of an investment paying 9% compounded monthly.

8. Compute precisely the tripling time of an investment paying 6% compounded continuously.

9. According to the CIA World Factbook, the population of Burkina Faso was 17,812,961 in July of 2013. The population is growing at a rate of 3.06% per year. If this rate were to remain constant, when will the population of Burkina Faso reach 20 million people? Please tell me both the month and the year.

10. If the growth rate of Burkina Faso from the previous problem remains constant, when will the population be triple its July of 2013 value? Please tell me both the month and the year.

11. If some country's population is growing at the rate of 2.1% per year, how long will it take the population to double? Please give the answer as a decimal in years, accurate to six significant figures.

12. In their 2014 annual report, (published in January), ConEdison claims that they have 3.3 million customers. If their customer base is growing very smoothly at 2% per year, when will they reach 4 million customers? Please tell me both the month and the year.

13. Repeat the above problem if the growth rate were 2.5% per year.

14. Warning: this one is a bit difficult. Suppose a quantity of radioactive material has 39,895 g when first found, and exactly one year later, has 39,817 g. What is the half-life of this material? (I'm going to use nine significant figures for all intermediate and final steps in my solution. Your results might vary if you use fewer digits of precision.)

15. You've already learned how to compute the cost-per-thousand or compounding-factor for continuously compounded interest. Please verify that skill by doing (or redoing) checkerboard Problem 3-9-16, in "Module 3.9: The Number e and Continuously Compounded Interest."

16. A particular new humanoid fossil has been found by an archeologist. In a sample that should have 28.91 grams of Carbon-14, measurements show only 5.314 grams of Carbon-14. Recalling that the half-life of Carbon-14 is 5730 years, how many years ago did the humanoid die?

17. In 2014, an excavation is being done in England and a medieval knight buried in his armor is found. A sample from the skull is made, and should have 78.75 grams of Carbon-14, but actually has 70.22 grams of Carbon-14. Recalling that the half-life of Carbon-14 is 5730 years, in what year did the knight die?

18. When a nuclear weapon is detonated, there are seven "long-term fission products" that remain behind for an extreme length of time. One of those is Iodine-129, and in "Module 3.6: Radiation, Bacteria, Population, and Real Estate" we learned that Iodine-129 has a half-life of 15.7 million years. Suppose one small section of dirt at the Nevada nuclear test site has 100 g of this isotope. How long will it be until only 10 g remain? (Hint: your unit for time and the half-life should be "millions of years," not "years.")

19. An important new disease is being studied in the lab. Two observations, made 25 minutes apart, indicate that a sample had 450,000 bacteria and 700,000 bacteria. What is the doubling time of this bacteria?

Answers

1. See my electronic online textbook-in-progress, near the citation given.

2. See my electronic online textbook-in-progress, near the citation given.

3. See my electronic online textbook-in-progress, near the citation given.

4. See my electronic online textbook-in-progress, near the citation given.

5. We begin with the compound interest formula. For doubling times, you can use any P you want, so long as $A = 2P$, but I prefer to use one million dollars. We can compute

$$i = r/m = 0.078/52 = 0.0015$$

and then proceed as follows:

$$\begin{aligned} A &= P(1+i)^n \\ 2,000,000 &= 1,000,000(1+0.0015)^n \\ 2 &= (1.0015)^n \\ \log 2 &= \log(1.0015)^n \end{aligned}$$

$$
\begin{aligned}
\log 2 &= n \log 1.0015 \\
\frac{\log 2}{\log 1.0015} &= n \\
462.444\cdots &= n
\end{aligned}
$$

Then we can see that 462.444 weeks is 8 years and $46.4\cdots$ weeks. Alternatively, you can use $n = 52t$ and then you get $t = 8.89316\cdots$, which is also 8 years and $46.4\cdots$ weeks.

6. The Rule of 72 predicts $72/7.8 = 9.23076\cdots$. That's a residual error of $0.337603\cdots$ years which is a relative error of $3.79617\cdots\%$. As you can see, the Rule of 72 is moderately good, but not all that great. Yet, regardless of its inaccuracy, I have to include because the other textbooks include it.

7. We begin with the compound interest formula, but use $A = 3,000,000$ and $P = 1,000,000$. Actually, just like we saw two problems ago, you can use any P so long as $A = 3P$. Next, we have

$$i = r/m = 0.09/12 = 0.0075$$

and so we can begin:

$$
\begin{aligned}
A &= P(1+i)^n \\
3,000,000 &= 1,000,000(1+0.0075)^{12t} \\
3 &= (1.0075)^{12t} \\
\log 3 &= \log(1.0075)^{12t} \\
\log 3 &= (12t)\log 1.0075 \\
\frac{\log 3}{\log 1.0075} &= 12t \\
147.030\cdots &= 12t \\
12.2525\cdots &= t
\end{aligned}
$$

Therefore we learn that the tripling time will be $12.2525\cdots$ years, or 12 years and 3.03 months. Be sure to write down a number accurate to six or more significant figures first, and only then round off on the next line.

8. We begin with the formula for continuously compounded interest. We could use any P we want, so long as $A = 3P$, but I'd like to use $P = 1,000,000$ and $A = 3,000,000$.

$$
\begin{aligned}
A &= Pe^{rt} \\
3,000,000 &= 1,000,000e^{0.06t} \\
3 &= e^{0.06t} \\
\ln 3 &= \ln e^{0.06t} \\
\ln 3 &= 0.06t \\
1.09861\cdots &= 0.06t \\
18.3102\cdots &= t
\end{aligned}
$$

Therefore, we learn that the investment will require $18.3102\cdots$ years to triple.

9. We begin with the standard formula for population:

$$\begin{aligned}
P(t) &= P_0(1+r)^t \\
20,000,000 &= (17,812,961)(1.0306)^t \\
1.12277\cdots &= (1.0306)^t \\
\ln 1.12277\cdots &= \ln(1.0306)^t \\
\ln 1.12277\cdots &= t\ln(1.0306) \\
\frac{\ln 1.12277}{\ln 1.0306} &= t \\
3.84211\cdots &= t
\end{aligned}$$

The equations predict that Burkina Faso's population will double in 3.84211 years, which is 3 years and $10.1054\cdots$ months. Exactly 3 years would be July of 2016, but then we have to move forward 10 more months. Therefore the answer is May of 2017.

10. Once again, we begin with the standard formula for population:

$$\begin{aligned}
P(t) &= P_0(1+r)^t \\
53,438,883 &= 17,812,961(1+0.0306)^t \\
3 &= (1.0306)^t \\
\ln 3 &= \ln(1.0306)^t \\
\ln 3 &= t\ln(1.0306) \\
\frac{\ln 3}{\ln 1.0306} &= t \\
36.4489\cdots &= t
\end{aligned}$$

The country will be triple its current population in $36.4489\cdots$ years, which is 36 years and $5.38690\cdots$ months. Just the 36 years brings us to July of 2050, but we add five months to obtain December of the year 2050.

11. Yet again, we begin with the standard formula for population:

$$\begin{aligned}
P(t) &= P_0(1+r)^t \\
2P_0 &= P_0(1+0.021)^t \\
2 &= (1+0.021)^t \\
2 &= (1.021)^t \\
\log 2 &= \log(1.021)^t
\end{aligned}$$

$$\begin{aligned}
\log 2 &= t\log(1.021)\\
\frac{\log 2}{\log 1.021} &= t\\
33.3523\cdots &= t
\end{aligned}$$

We conclude that the population of this particular country will double in $33.3523\cdots$ years.

12. We should just pretend that the number of customers of ConEdison is just the population of some country. If we do so, then we obtain the following calculation:

$$\begin{aligned}
P(t) &= P_0(1+r)^t\\
P(t) &= (3,300,000)(1.02)^t\\
4,000,000 &= (3,300,000)(1.02)^t\\
\frac{4,000,000}{3,300,000} &= 1.02^t\\
1.21\overline{21} &= 1.02^t\\
\log 1.21\overline{21} &= \log 1.02^t\\
0.0835460\cdots &= t\log 1.02\\
0.0835460\cdots &= t(0.00860017\cdots)\\
\frac{0.0835460\cdots}{0.00860017\cdots} &= t\\
9.71446\cdots &= t
\end{aligned}$$

The value of $9.71446\cdots$ comes out to be 9 years, 8 months, and $17.2067\cdots$ days using 12 months per year and 30 days per month. You might recall that this notion is called "Banker's Rule." In any case, it is clear that the answer is that ConEdison will reach 4 million customers in September of 2023.

Let's check now:

$$(3,300,000)(1.02)^{9.71446} = (3,300,000)(1.21212\cdots) = 3,999,999.\cdots$$

13. Using the methods of the previous problem, we obtain $t = 7.79066\cdots$. We can check with

$$(3,300,000)(1.025)^{7.79066} = (3,300,000)(1.21212\cdots) = 3,999,999.\cdots$$

and thus we know our answer is $t = 7.79066$. That is equivalent to 7 years, 9 months, and 14.6396 days. Therefore, it is clear that the answer is that ConEdison will reach 4 million customers in October of 2021.

14. We begin by taking the radioactive decay formula, plugging in the known values, and solving for H.

$$\begin{aligned}
A(t) &= A_0 2^{-t/H} \\
39,817 &= (39,895)2^{-1/H} \\
0.998044867\cdots &= 2^{-1/H} \\
\log 0.998044867\cdots &= \log 2^{-1/H} \\
\log 0.998044867\cdots &= \frac{-1}{H}\log 2 \\
\frac{\log 0.998044867\cdots}{\log 2} &= \frac{-1}{H} \\
-0.00282342054\cdots &= \frac{-1}{H} \\
\frac{1}{-0.00282342054\cdots} &= -H \\
-354.180323\cdots &= -H \\
354.180323\cdots &= H
\end{aligned}$$

Thus we learn that the half-life of this unknown substance is 354 years. I think it is very significant that you reflect on this. We've established that fact mathematically, using 1 year of observation rather than 354 years. That's good, because 354 years before 2014 would be 1660, and human society has known about radioactive decay for that long. Critics of archeology, paleontology, and evolutionary biology often criticize radiocarbon dating because the half-life of Carbon-14 is 5730 years—yet no one has been making radioactive measurements for 5730 years. Now you can see how half-lives can be computed mathematically, without requiring a chemist to sit waiting for 354 or 5730 years.

15. See my electronic online textbook-in-progress, near the citation given.

16. Naturally, we begin with the standard radioactivity formula, and plug in the given numbers.

$$\begin{aligned}
A(t) &= A_0 2^{-t/H} \\
5.314 &= 28.912^{-t/5730} \\
\frac{5.314}{28.91} &= 2^{-t/5730} \\
0.183811\cdots &= 2^{-t/5730} \\
\ln 0.183811\cdots &= \ln 2^{-t/5730} \\
-1.69384\cdots &= \frac{-t}{5730}\ln 2 \\
-1.69384\cdots &= \frac{-t}{5730}(0.693147\cdots) \\
\frac{(-1.69384\cdots)(5730)}{(0.693147\cdots)} &= -t
\end{aligned}$$

$$\begin{aligned} -14,002.3\cdots &= -t \\ 14,002.3\cdots &= t \end{aligned}$$

As you can see, this fossil is extremely recent, being approximately 14,002.3 years old. Alternatively, one can say that it is from the year 11,988 BCE.

17. Using the techniques of the previous problem, you will get that $t = 947.730\cdots$. Therefore, we might be tempted to say that the knight died 947.730 years ago. A close re-reading of the question reveals that this is not what we were asked. Instead, we were asked to compute the year of the knight's death. Our calculations do not mean that the knight died in the year 947, however. Since the knight died 947.730 years before the excavation in 2014, we calculate

$$2014 - 947.730 = 1066.27$$

and so we know that the knight died in the year 1066, presumably at The Battle of Hastings.

18. We can just start with the standard setup of a half-life problem, because all the data we need has been given, except for t.

$$\begin{aligned} A(t) &= (A_0)2^{-t/H} \\ 10 &= (100)2^{-t/15.7} \\ 10/100 &= 2^{-t/15.7} \\ 0.1 &= 2^{-t/15.7} \\ \log 0.1 &= \log 2^{-t/15.7} \\ -1 &= \frac{-t}{15.7}\log 2 \\ -1 &= \frac{-t}{15.7}(0.301029\cdots) \\ \frac{-15.7}{0.301029\cdots} &= -t \\ -52.15427\cdots &= t \\ 52.1542\cdots &= t \end{aligned}$$

Therefore, our answer is 52.1542 million years. Note, it would be phenomenally wrong to write 52.1542 years. The word "million" is important there. In any case, this is why we should not take the use of nuclear weapons lightly.

19. Of course, for bacteria, the basic formula always is

$$P(t) = P_0 2^{t/D}$$

where P_0 is the initial condition, t is time, and D is the doubling time. The initial condition would be the 450,000 bacteria. We don't know the doubling time, so we'll call it D minutes. At the moment we have

$$P(t) = (450,000)2^{t/D}$$

which doesn't look like terribly much progress. However, we know that $P(25) = 700,000$. This allows us to write

$$\begin{aligned}
700,000 &= P(25) \\
700,000 &= (450,000)2^{25/D} \\
\frac{700,000}{450,000} &= 2^{25/D} \\
1.5\overline{5} &= 2^{25/D} \\
\log 1.5\overline{5} &= \log 2^{25/D} \\
0.191885\cdots &= \frac{25}{D}\log 2 \\
0.191885\cdots &= \frac{25}{D}(0.3010299\cdots) \\
(0.191885\cdots)(0.3010299\cdots) &= \frac{25}{D} \\
0.637429\cdots &= \frac{25}{D} \\
(0.637429\cdots)D &= 25 \\
D &= \frac{25}{0.637429\cdots} \\
D &= 39.2199\cdots
\end{aligned}$$

Therefore, we conclude that the half-life is $39.2199\cdots$ minutes, which would almost surely be rounded to 39.2 minutes in practice. Meanwhile, we must check our work with the unrounded value.

$$(450,000)2^{25/39.2199} = (450,000)2^{0.637431\cdots} = (450,000)(1.55555\cdots) = 700,000.\cdots$$